MODULES OVER TRIANGULATED CATEGORIES AND LOCALIZATION

GEORGE CIPRIAN MODOI

Abstract. A (right) module over a preadditive category is an additive contravariant functor defined on it with values in the category of abelian groups. We show, for a compactly generated triangulated category, that the category of modules over its subcategory consisting of all compact objects it is not only the colocalization, but also the localization of the category of finitely presented modules over the full triangulated category.

How was observed by many authors, a difficult problem which arises in the study of triangulated categories is that the axioms which define them do not have a functorial character. Thus, it is not unimportant to find some abelian categories closely related to a given triangulated one. A such category is the category of finitely presented contravariant functors defined on it with values in the category of abelian groups. We denote it here by $\mathrm{mod}(\mathcal{T})$, where $\mathcal{T}$ is the triangulated category. The Yoneda embedding gives an universal homological functor $\mathbf{h} : \mathcal{T} \to \mathrm{mod}(\mathcal{T})$ [3, 5.1.18]. A result due to Neeman [3, 5.3.9] says, that a triangulated functor between two triangulated categories $\mathcal{T} \to \mathcal{S}$ have a right or a left adjoint if and only if the induced functor $\mathrm{mod}(\mathcal{T}) \to \mathrm{mod}(\mathcal{S})$ does. But it is also not easy to deal with the category $\mathrm{mod}(\mathcal{T})$, since it may be not well-powered [3, Appendix C], in the sense that an object may have a proper class (which is not a set) of subobjects (quotients). In the same work of Neeman [3], was observed that a "good" approximation of the category $\mathrm{mod}(\mathcal{T})$ is the category $\mathrm{Ex}((\mathcal{T}^{\alpha})^{\mathrm{op}}, \mathcal{A}b)$, whose objects are additive functors $(\mathcal{T}^{\alpha})^{\mathrm{op}} \to \mathcal{A}b$ which take coproducts fewer than α objects in products in $\mathcal{A}b$. Here α is a fixed regular cardinal, $\mathcal{T}^{\alpha}$ is the full subcategory of α-compact objects of $\mathcal{T}$, in the sense of the definition [3, 4.2.7], and it is supposed to be skeletally small. Precisely, the category $\mathrm{Ex}((\mathcal{T}^{\alpha})^{\mathrm{op}}, \mathcal{A}b)$ is the colocalization of $\mathrm{mod}(\mathcal{T})$ [3, 6.5.3]. In this note we prove, for $\mathcal{C} = \mathcal{T}^{\aleph_0}$ i.e. $\alpha = \aleph_0$, so $\mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b) = \mathrm{Mod}(\mathcal{C})$ contains all functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{A}b$, that $\mathrm{Mod}(\mathcal{C})$ is not only the colocalization, but also the localization of $\mathrm{mod}(\mathcal{T})$.

A few words about terminology and notations: By $\mathcal{A}b$ we shall denote the category of abelian groups. We shall write $\mathcal{A} \to \mathcal{B}$ respectively $\mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ to emphasize that we deal with a covariant (contravariant) functor between two given categories $\mathcal{A}$ and $\mathcal{B}$. It is well-known that an associative ring R may

Key words and phrases. triangulated category, module over an preadditive category, compact object, localization, colocalization.

be regarded as a preadditive category with a single object, and then a right R-module means a functor $R^{\mathrm{op}} \to \mathcal{A}b$. The additive functors $\mathcal{A}^{\mathrm{op}} \to \mathcal{A}b$, defined on an arbitrary preadditive category $\mathcal{A}$ will also call (right) modules over $\mathcal{A}$, or simply $\mathcal{A}$-modules. We denote by $\mathcal{A}(a', a)$ and $\mathrm{Hom}_{\mathcal{A}}(M', M)$ the set of morphism between objects a' and a, in the category $\mathcal{A}$, respectively the class of all natural transformations between $\mathcal{A}$-modules M' and M.

For basic facts about abelian categories, we refer the reader to [4], and for the general theory of triangulated categories to [3]. Even if in the text all references concerning abelian categories are to [4], the personal experience of the author playing a role here, this things may be found also in many works, for example in Gabriel's [1].

Let $\mathcal{T}$ be a triangulated category, and $\mathcal{C}$ its full subcategory consisting of all compact objects. Recall that an object $c \in \mathcal{T}$ is called *compact* provided that the covariant functor $\mathcal{T}(c, -) : \mathcal{T} \to \mathcal{A}b$ commutes with direct sums. It is well-known, and also easy to see, that $\mathcal{C}$ is a *thick* subcategory of $\mathcal{T}$, that means, a triangulated subcategory which closed under direct summands. Throughout of this note we assume $\mathcal{T}$ has arbitrary coproducts, $\mathcal{C}$ is a skeletally small category, and it generates $\mathcal{T}$, i.e. $\mathcal{T}(c, x) = 0$ for all $c \in \mathcal{C}$ implies $x = 0$.

By a (right) module over $\mathcal{C}$, we understand, as in the case of ordinary modules over a ring, an additive contravariant functor $M : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}b$. The class of all modules over $\mathcal{C}$ together with the natural transformations between them, form a Grothendieck category, denoted by $\mathrm{Mod}(\mathcal{C})$ [4, chapter 4, 4.9], where the limits and the colimits are computed pointwise. A module N over the category $\mathcal{T}$ is called *finitely presented*, if there is an exact sequence of functors and natural transformations

$$\mathcal{T}(-, s) \to \mathcal{T}(-, t) \to N \to 0.$$

By [3, 5.1.10], the class $\mathrm{mod}(\mathcal{T})$ of all finitely presented modules over $\mathcal{T}$ together with natural transformations between them forms an abelian category. Note that, even if the class of all modules over $\mathcal{T}$ forms only a illegitimate category, in the sense that the class of the natural transformations between two such modules may be proper, this does not happen with $\mathrm{mod}(\mathcal{T})$, it being a good defined category [3, 5.1.15].

The functor $\mathbf{h} : \mathcal{T} \to \mathrm{mod}(\mathcal{T})$, $\mathbf{h}(t) = \mathcal{T}(-, t)$ is a homological embedding, which send any object t of $\mathcal{T}$ to a projective object of $\mathrm{mod}(\mathcal{T})$. Moreover, since $\mathcal{T}$ is idemsplit (that is every idempotent $t \to t$ splits, for all $t \in \mathcal{T}$) [3, 1.6.8], every projective object of $\mathrm{mod}(\mathcal{T})$ is of this form [3, 5.1.11]. Restricting to $\mathcal{C}$ the images of $\mathbf{h}$ on each object $t \in \mathcal{T}$, we obtain a homological functor $\bar{\mathbf{h}} : \mathcal{T} \to \mathrm{Mod}(\mathcal{C})$, $\bar{\mathbf{h}}(t) = \mathcal{T}(-, t)|_{\mathcal{C}}$. Clearly $\bar{\mathbf{h}}$ commutes with coproducts, and for any $M \in \mathrm{Mod}(\mathcal{C})$, there is an exact sequence

$$\begin{tikzcd}
\bigoplus_{j\in J}\bar{\mathbf{h}}(d_j) \arrow[r] \arrow[d, equal] & \bigoplus_{i\in I}\bar{\mathbf{h}}(c_i) \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, equal] & 0 \\
\bar{\mathbf{h}}(\bigoplus_{j\in J} d_j) \arrow[r] & \bar{\mathbf{h}}(\bigoplus_{i\in I}\bar{\mathbf{h}}(c_i)) \arrow[r] & M \arrow[r] & 0,
\end{tikzcd}$$

with d_j and c_i belonging to $\mathcal{C}$.

Since $\mathbf{h}$ is an universal homological functor [3, 5.1.18], it results an exact functor $\boldsymbol{\pi}$ making commutative the diagram

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "\mathbf{h}"] \arrow[dr, "\bar{\mathbf{h}}"] & \mathrm{mod}(\mathcal{T}) \arrow[d, "\boldsymbol{\pi}"] \\
& \mathrm{Mod}(\mathcal{C}).
\end{tikzcd}$$

Because every additive contravariant functor takes finite coproducts to products, we lie in the hypothesis of [3, chapter 6]. It follows that $\boldsymbol{\pi}(M) = M|_{\mathcal{C}}$ [3, 6.5.2], and $\mathrm{Mod}(\mathcal{C})$ is the colocalization of $\mathrm{mod}(\mathcal{T})$, what means, the functor $\boldsymbol{\pi}$ has a fully-faithful left adjoint $\mathbf{L} : \mathrm{Mod}(\mathcal{C}) \to \mathrm{mod}(\mathcal{T})$. This adjoint is determined by its right exactness, and by the equality $\mathbf{L}(\bigoplus_{i\in I}\bar{\mathbf{h}}(c_i)) = \mathbf{h}(\bigoplus_{i\in I}\bar{\mathbf{h}}(c_i))$, for all $c_i \in \mathcal{C}$ [3, 6.5.3]. Denote by $v : 1_{\mathrm{Mod}(\mathcal{C})} \to \boldsymbol{\pi}\mathbf{L}$ and $u : \mathbf{L}\boldsymbol{\pi} \to 1_{\mathrm{mod}(\mathcal{T})}$ the unit, respectively the counit, of this adjunction. It is well-known that the fully-faithfulness of $\mathbf{L}$ is equivalent to the existence of an inverse for v [4, chapter 1, 13.11].

Lemma 1. *Any projective object P of* $\mathrm{Mod}(\mathcal{C})$ *is of the form, $\bar{\mathbf{h}}(c)$ for an object $c = \bigoplus_{i\in I} c_i \in \mathcal{T}$, with $c_i \in \mathcal{C}$, and the induced map*

$$\mathcal{T}(c, x) \to \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(c), \bar{\mathbf{h}}(x))$$

is an isomorphism for all $x \in \mathcal{T}$.

Proof. A projective object P of $\mathrm{Mod}(\mathcal{C})$ is a direct summand of a direct sum $\bigoplus_{j\in J}\mathcal{C}(-, d_j)$, and because $\mathcal{C}$ is idemsplit, it follows $P \cong \bar{\mathbf{h}}(\bigoplus_{i\in I} c_i) = \bar{\mathbf{h}}(c)$.

Using the isomorphism of adjunction, and then the Yoneda isomorphism, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(c), \bar{\mathbf{h}}(x)) &= \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(c), \boldsymbol{\pi}(\mathbf{h}(x))) \cong \mathrm{Hom}_{\mathcal{T}}(\mathbf{L}(\bar{\mathbf{h}}(c)), \mathbf{h}(c)) \\ &\cong \mathrm{Hom}_{\mathcal{T}}(\mathbf{h}(c), \mathbf{h}(x)) \cong \mathcal{T}(c, x).\end{aligned}$$

□

We record also an analogous for injectives:

Lemma 2. [2, Lemma 1] *Any injective object Q of* $\mathrm{Mod}(\mathcal{C})$ *is of the form $\bar{\mathbf{h}}(q)$, for an object $q \in \mathcal{T}$, and the induced map*

$$\mathcal{T}(x, q) \to \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(x), \bar{\mathbf{h}}(q))$$

is an isomorphism for all $x \in \mathcal{T}$.

Lemma 3. *The assignement $M \mapsto \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(-), M)$ gives a functor*

$$\mathbf{R} : \mathrm{Mod}(\mathcal{C}) \to \mathrm{mod}(\mathcal{T}).$$

Proof. The unique problem which arises is that $\mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(-), M) : \mathcal{T}^{\mathrm{op}} \to \mathcal{A}b$ is actually finitely presented, for all $M \in \mathrm{Mod}(\mathcal{C})$.

Choose an injective resolution for M:

$$0 \to M \to Q_1 \to Q_2.$$

Fix an object $x \in \mathcal{T}$. Applying the left exact functor $\mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(x), -)$ to this injective resolution, and using 2, it follows that there are two objects $q_1, q_2 \in \mathcal{T}$, and a commutative diagram of abelian groups:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(x), M) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(x), Q_1) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(x), Q_2) \arrow[d, equal] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(x), M) \arrow[r] & \mathcal{T}(x, q_1) \arrow[r] & \mathcal{T}(x, q_2).
\end{tikzcd}$$

Therefore $\mathrm{Hom}(\bar{\mathbf{h}}(-), M)$ is pointwise the kernel of the natural transformation $\mathcal{T}(-, q_1) \to \mathcal{T}(-, q_2)$ between two finitely presented $\mathcal{T}$-modules. Then, by [3, 5.1.10], this functor belongs to $\mathrm{mod}(\mathcal{T})$. $\square$

Now we are ready to give the main result of this note.

Theorem 4. *The functor $\mathbf{R} : \mathrm{Mod}(\mathcal{C}) \to \mathrm{mod}(\mathcal{T})$ is the fully-faithful right adjoint of the functor $\boldsymbol{\pi} : \mathrm{mod}(\mathcal{T}) \to \mathrm{Mod}(\mathcal{C})$, so the category $\mathrm{Mod}(\mathcal{C})$ is not only the colocalization, but also the localization of the category $\mathrm{mod}(\mathcal{T})$.*

Proof. Let $c \in \mathcal{C}$, and $M : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}b$ be a $\mathcal{C}$-module. Then, the Yoneda isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(c), M) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{T}(-, c)|_{\mathcal{C}}, M) \cong \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}(-, c), M) \cong M(c)$$

shows that $\boldsymbol{\pi}\mathbf{R}(M) = \mathbf{R}(M)|_{\mathcal{C}} = \mathrm{Hom}(\bar{\mathbf{h}}(-), M)|_{\mathcal{C}}$ is naturally isomorphic to M. Denote by $v' : \boldsymbol{\pi}\mathbf{R} \to 1_{\mathrm{Mod}(\mathcal{C})}$ this isomorphism.

Let now $N : \mathcal{T}^{\mathrm{op}} \to \mathcal{A}b$ be a finitely presented $\mathcal{T}$-module. Then we have

$$\begin{aligned}
\mathbf{R}\boldsymbol{\pi}(N) &= \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(-), \boldsymbol{\pi}(N)) \cong \mathrm{Hom}_{\mathcal{T}}(\mathbf{L}(\bar{\mathbf{h}}(-)), N) \\
&= \mathrm{Hom}_{\mathcal{T}}(\mathbf{L}\boldsymbol{\pi}(\mathbf{h}(-)), N),
\end{aligned}$$

and again an Yoneda isomorphism

$$\mathrm{Hom}_{\mathcal{T}}(\mathbf{h}(-), N) \cong N.$$

The counit $u_{\mathbf{h}(-)} : \mathbf{L}\boldsymbol{\pi}(\mathbf{h}(-)) \to \mathbf{h}(-)$ of the adjunction between $\mathbf{L}$ and $\boldsymbol{\pi}$ gives a morphism

$$u'_N = \mathrm{Hom}_{\mathcal{T}}(u_{\mathbf{h}(-)}, N) : N \cong \mathrm{Hom}_{\mathcal{T}}(\mathbf{h}(-), N) \to \mathrm{Hom}_{\mathcal{T}}(\mathbf{L}\boldsymbol{\pi}(\mathbf{h}(-)), N) \cong \mathbf{R}\boldsymbol{\pi}(N),$$

so a natural transformation $u' : 1_{\mathrm{mod}(\mathcal{T})} \to \mathbf{R}\boldsymbol{\pi}$.

Fix $c \in \mathcal{C}$, $t \in \mathcal{T}$, $M \in \mathrm{Mod}(\mathcal{C})$ and $N \in \mathrm{mod}(\mathcal{T})$. The maps $\mathbf{R}(v'_M)$, $v'_{\boldsymbol{\pi}(N)}$ are clearly isomorphisms since v' is so. Moreover, the maps

$$(\boldsymbol{\pi}(u'_N))_c : \boldsymbol{\pi}(N) = N(c) \to \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(c), N|_{\mathcal{C}})$$

and

$$(u'_{\mathbf{R}(M)})_t : \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(t), M) \to \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(t), \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(-), M)|_{\mathcal{C}})$$

are isomorphisms too, by an analogous argument to the one used for v. Hence, the naturality of these morphisms implies the equalities $\mathbf{R}(v'_M)u'_{\mathbf{R}(M)} = 1_{\mathbf{R}(M)}$ and $v'_{\boldsymbol{\pi}(N)}\boldsymbol{\pi}(u'_N) = 1_{\boldsymbol{\pi}(N)}$, which show that $\mathbf{R}$ is the right adjoint of $\boldsymbol{\pi}$, with the unit u' and the counit v'.

Finally the fully-faithfulness of $\mathbf{R}$ is equivalent, by [4, chapter 1, 13.10], to the fact that v' is invertible. □

Remark 5. The subcategory $\mathrm{Ker}\,\boldsymbol{\pi}$ of $\mathrm{mod}(\mathcal{T})$, consisting of the objects sended by $\boldsymbol{\pi}$ to 0 is both localizing and colocalizing, and the categories $\mathrm{mod}(\mathcal{T})/\,\mathrm{Ker}\,\boldsymbol{\pi}$, $\mathrm{Mod}(\mathcal{C})$ and $\mathrm{Ker}\,\boldsymbol{\pi}\backslash\mathrm{mod}(\mathcal{T})$ are all equivalent.

References

[1] P. Gabriel, *De catégories abeliennes*, Bull. Soc. Math. France, 90(1962) 323-448.
[2] H. Krause, *Brown representability and flat covers*, J. Pure and Appl. Alg., 157(2001) 81-86.
[3] A. Neeman, *Triangulated categories*, ????.
[4] N. Popescu, L. Popescu, *Theory of categories*, Editura Academiei, Bucureşti, România and Sijthoff & Noordhoff International Publishers, 1979.

Babeş–Bolyai University, Faculty of Mathematics and Computer Science, 1, Mihail Kogălniceanu, 400084 Cluj–Napoca, Romania
E-mail address: cmodoi@math.ubbcluj.ro